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# POTENTIAL THEORETIC APPROACH TO SCHAUDER ESTIMATES FOR THE FRACTIONAL LAPLACIAN

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**ABSTRACT.** We present an elementary approach for the proof of Schauder estimates for the equation  $(-\Delta)^s u(x) = f(x)$ ,  $0 < s < 1$ , with  $f$  having a modulus of continuity  $\omega_f$ , based on the Poisson representation formula and dyadic ball approximation argument. We give the explicit modulus of continuity of  $u$  in balls  $B_r(x) \subset \mathbb{R}^n$  in terms of  $\omega_f$ .

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## 1. INTRODUCTION

Let  $f$  be a given Hölder continuous function and  $u$  solving  $(-\Delta)^s u(x) = f(x)$  with a fixed  $s \in (0, 1)$ . We want to study the regularity of  $u$  using a very simple method based on the Poisson representation formula and dyadic ball approximation argument. In order to formulate our results it is convenient to introduce some notations and basic knowledge on the fractional Laplacian and on some related kernels. For further details on the fractional Laplacian and applications, see [2, 4]. Also, for regularity up to the boundary of weak solutions of the Dirichlet problem, see the very nice paper [11].

In what follows we assume that  $n \geq 2$ . We denote by  $\mathcal{S}$  the Schwartz space of rapidly decreasing functions, defined as follows

$$\mathcal{S} = \left\{ u \in C^\infty(\mathbb{R}^n) \quad \text{s.t. for any } \alpha, \beta \in \mathbb{N}_0^n, \quad \sup_{x \in \mathbb{R}^n} |x^\beta D^\alpha u(x)| < \infty \right\}.$$

Let  $s \in (0, 1)$  be fixed. We have the following integral definition.

**Definition 1.1.** *The fractional Laplacian of  $u \in \mathcal{S}$  is defined for any  $x \in \mathbb{R}^n$  as*

$$\begin{aligned} (-\Delta)^s u(x) &= C(n, s) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(x-y)}{|y|^{n+2s}} dy \\ &= C(n, s) \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon} \frac{u(x) - u(x-y)}{|y|^{n+2s}} dy, \end{aligned}$$

where P.V. stand for “in the principal value sense”, as defined in the last line, and  $C(n, s)$  is a constant depending only on  $n$  and  $s$ . In what follows we call such constants dimensional.

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With a change of variables, one obtains an equivalent representation, given by

$$(-\Delta)^s u(x) = \frac{C(n, s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x-y) - u(x+y)}{|y|^{n+2s}} dy.$$

In this expression the principle value may be omitted if  $u \in L_s^1(\mathbb{R}^n) \cap C^{2s+\varepsilon}(B_r(x))$  for a small  $r > 0$ . Here, the space  $L_s^1(\mathbb{R}^n)$  is the weighted  $L^1$  space, defined as

$$L_s^1(\mathbb{R}^n) := \left\{ u \in L_{\text{loc}}^1(\mathbb{R}^n) \text{ s.t. } \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < \infty \right\}.$$

Moreover, we denote by  $C^{2s+\varepsilon}$  for small  $\varepsilon$  the Hölder space  $C^{0,2s+\varepsilon}$  for  $s < 1/2$  and  $C^{1,2s+\varepsilon-1}$  for  $s \geq 1/2$ .

The main result we prove in this paper is a Schauder type estimate for bounded solutions of the equation  $(-\Delta)^s u = f$  in  $B_1$ . Given  $f \in C^{0,\alpha}(B_1) \cap C(\overline{B_1})$ , then on the half ball  $u$  has the regularity of  $f$  increased by  $2s$ . More precisely:

**Theorem 1.2.** *Let  $s \in (0, 1)$ ,  $\alpha < 1$  and  $f \in C^{0,\alpha}(B_1) \cap C(\overline{B_1})$  be a given function with modulus of continuity  $\omega(r) := \sup_{|x-y|<r} |f(x) - f(y)|$ . Let  $u \in L^\infty(\mathbb{R}^n) \cap C^1(B_1)$  be a pointwise solution of  $(-\Delta)^s u = f$  in  $B_1$ . Then for any  $x, y \in B_{1/2}$  and denoting  $\delta := |x - y|$  we have that for  $s \leq 1/2$*

$$|u(x) - u(y)| \leq C_{n,s} \left( \delta \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)} + \delta \sup_{\overline{B_1}} |f| + \int_0^{c\delta} \omega(t) t^{2s-1} dt + \delta \int_\delta^1 \omega(t) t^{2s-2} dt \right) \quad (1.1)$$

while for  $s > 1/2$

$$|Du(x) - Du(y)| \leq C_{n,s} \left( \delta \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)} + \delta \sup_{\overline{B_1}} |f| + \int_0^{c\delta} \omega(t) t^{2s-2} dt + \delta \int_\delta^1 \omega(t) t^{2s-3} dt \right), \quad (1.2)$$

where  $C_{n,s}$  and  $c$  are positive dimensional constants.

There are other approaches to prove Schauder estimates for the fractional order operators with more general kernels see [6] and references therein. Here we follow the one proposed by Xu-Jia Wang in [14] which is based only on the higher order derivative estimates, that we state here in Lemma 3.1 and on a maximum principle, given in Lemma 3.4.

We prove these estimates using some kernels related to the fractional Laplacian (see Chapter I.6 in [10] or [1] for more details), that we introduce now. We take  $r > 0$  and introduce the fractional Poisson kernel on the ball. For any  $x \in B_r$  and any  $y \in \mathbb{R}^n \setminus \overline{B_r}$  we define

$$P_r(y, x) := c(n, s) \left( \frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^s \frac{1}{|x - y|^n}, \quad (1.3)$$

where  $c(n, s)$  is a dimensional constant given in such a way that

$$\int_{\mathbb{R}^n \setminus B_r} P_r(y, x) dy = 1.$$

Then (see Theorem 2.10 in [1]) one has for  $u \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n \setminus B_r)$  that the equation

$$(-\Delta)^s u(x) = 0 \text{ in } B_r$$

has a pointwise solution given by

$$u(x) = \int_{\mathbb{R}^n \setminus B_r} P_r(y, x) u(y) dy. \quad (1.4)$$

We recall also a representation formula for the equation  $(-\Delta)^s u = f$  in  $\mathbb{R}^n$ . For any  $x \in \mathbb{R}^n \setminus \{0\}$  we define

$$\Phi(x) := a(n, s)|x|^{-n+2s}, \quad (1.5)$$

where  $a(n, s)$  is a dimensional constant. The function  $\Phi$  plays the role of the fundamental solution of the fractional Laplacian, i.e. in the distributional sense

$$(-\Delta)^s \Phi = \delta_0,$$

where  $\delta_0$  is the Dirac delta evaluated at 0 (see Theorem 2.3 in [1] for the proof). For a function  $f \in C_c^{0,\varepsilon}(\mathbb{R}^n)$ , where  $\varepsilon > 0$  is a small quantity, we define

$$u(x) := f * \Phi(x) = a(n, s) \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2s}} dy. \quad (1.6)$$

Then (see Lemma 2.6 in [1]) we have that  $u \in L_s^1(\mathbb{R}^n)$ . Moreover  $u \in C^{2s+\varepsilon}(\mathbb{R}^n)$  as we see in the next Section 2. This, together with Theorem 2.8 in [1] says that  $u$  is pointwise solution of

$$(-\Delta)^s u = f \quad \text{in } \mathbb{R}^n. \quad (1.7)$$

One of the motivations to study (1.7) comes from the active scalars (see [3]). The 2D incompressible Euler equation

$$\begin{cases} \omega_t + v \nabla \omega = 0 \\ v = (\partial_2 \psi, -\partial_1 \psi) \\ \omega = \Delta \psi \end{cases} \quad (1.8)$$

is one of the well-known active scalar equations. Here  $v$  is the velocity,  $\omega$  the vorticity,  $\psi$  the stream function.

The uniqueness was proved by Yudovich (see [15]) under the condition that  $\omega(t) \in L^\infty(0, T; L^1(\mathbb{R}^2)) \cap L^\infty(0, T; L^\infty(\mathbb{R}^2))$ . Observe that by the Biot-Savart law one has that  $v = k * \omega$ , where

$$k(x) = \frac{x^\perp}{2\pi|x|^2}.$$

Clearly  $k \in L_{loc}^p(\mathbb{R}^2)$ ,  $1 \leq p < 2$  and  $k \in L^q(\mathbb{R}^2)$ ,  $q > 2$  near infinity, implying that one must assume  $\omega \in L^{p_o}(\mathbb{R}^2) \cap L^{q_o}(\mathbb{R}^2)$ ,  $p_o < 2 < q_o$  to make sure that  $v = k * \omega$  is well defined. In particular  $p_o = 1$ ,  $q_o = \infty$  will do.

A generalization of the 2D Euler equation is the quasigeostrophic active scalar

$$\begin{cases} \omega_t + v \nabla \omega = 0 \\ v = (\partial_2 \psi, -\partial_1 \psi) \\ -\omega = (-\Delta)^{\frac{1}{2}} \psi \end{cases} \quad (1.9)$$

or more generally when one takes  $-\omega = (-\Delta)^s \psi$ ,  $0 < s < 1$ . Thus this leads to the study of  $k_s * (-\Delta)^{-s} \omega$  where  $n = 2$  and

$$k_s(x) = \nabla^\perp \frac{C_{n,s}}{|x|^{n-2s}}.$$

We see that the regularity of the stream function can be concluded from that of  $\omega$  via Schauder estimates.

## 2. HÖLDER ESTIMATES FOR THE RIESZ POTENTIALS

In the next Lemma, we establish that given a bounded function with bounded support, its convolution with the function  $\Phi$  defined in (1.5) is Hölder continuous.

**Lemma 2.1.** *Let  $s \in (0, 1/2) \cup (1/2, 1)$  be fixed. Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded set, the function  $f \in L^\infty(\mathbb{R}^n)$  be supported in  $\Omega$  and  $u$  be defined as*

$$u(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-2s}} dy. \quad (2.1)$$

Then  $u \in C^{0,2s}(\mathbb{R}^n)$  for  $s < 1/2$  and  $u \in C^{1,2s-1}$  for  $s > 1/2$ .

The proof of this Lemma takes inspiration from [15], where some bounds are obtained in the case  $s = 1/2$ . Check also Lemma 3.1 in [5] for other considerations.

*Proof.* Let  $s < 1/2$  be fixed. We consider  $x_1, x_2 \in \mathbb{R}^n$  and denote by  $\delta := |x_1 - x_2|$ . We notice that in the course of the proof, the constants may change value from line to line. We have that

$$\begin{aligned} & |u(x_1) - u(x_2)| \\ & \leq \int_{\Omega} |f(y)| \left| \frac{1}{|x_1 - y|^{n-2s}} - \frac{1}{|x_2 - y|^{n-2s}} \right| dy \\ & \leq \|f\|_{L^\infty(\mathbb{R}^n)} \left[ \int_{\Omega \cap \{|x_1 - y| \leq 2\delta\}} \frac{dy}{|x_1 - y|^{n-2s}} + \int_{\Omega \cap \{|x_1 - y| \leq 2\delta\}} \frac{dy}{|x_2 - y|^{n-2s}} \right. \\ & \quad \left. + \int_{\Omega \setminus \{|x_1 - y| \leq 2\delta\}} \left| \frac{1}{|x_1 - y|^{n-2s}} - \frac{1}{|x_2 - y|^{n-2s}} \right| dy \right] \\ & =: \|f\|_{L^\infty(\mathbb{R}^n)} (I_1 + I_2 + I_3). \end{aligned} \quad (2.2)$$

By passing to polar coordinates we have that

$$I_1 \leq C_n \int_0^{2\delta} \rho^{2s-1} d\rho = C_{n,s} \delta^{2s}.$$

At the same manner, noticing that  $|x_2 - y| \leq |x_2 - x_1| + |x_1 - y| \leq 3\delta$  we have

$$I_2 \leq C_n \int_0^{3\delta} \rho^{2s-1} d\rho = C_{n,s} \delta^{2s}.$$

For  $I_3$ , we see that  $|x_2 - y| \geq |x_1 - y| - |x_1 - x_2| \geq \delta$ . The function  $|x - y|^{2s-n}$  is differentiable at each point of the segment  $x_1 x_2$  and using the mean value theorem we have that for  $x^*$  on the segment  $x_1 x_2$

$$\left| \frac{1}{|x_1 - y|^{n-2s}} - \frac{1}{|x_2 - y|^{n-2s}} \right| \leq C_n \frac{|x_1 - x_2|}{|x^* - y|^{n-2s+1}} = C_n \delta \frac{1}{|x^* - y|^{n-2s+1}}.$$

It follows that

$$I_3 \leq C_n \delta \int_{\Omega \setminus \{|x_1 - y| \leq 2\delta\}} \frac{1}{|x^* - y|^{n-2s+1}} dy.$$

Since  $|x^* - x_1| \leq \delta \leq \frac{1}{2}|x_1 - y|$ , we have that  $|x^* - y| \geq \frac{1}{2}|x_1 - y|$ . Passing to polar coordinates, since  $2s < 1$ , we get that

$$I_3 \leq C_n \delta \int_{2\delta}^{\infty} \rho^{2s-2} d\rho = C_{n,s} \delta^{2s}. \quad (2.3)$$

By inserting these bounds into (2.2) we obtain that

$$|u(x_1) - u(x_2)| \leq C \delta^{2s}, \quad (2.4)$$

where  $C = C(n, s, f)$  is a positive constant. To prove the bound for  $s > 1/2$ , thanks to Lemma 4.1 in [9] we have that

$$Du(x) = \int_{\Omega} D\Phi(x-y)f(y)dy = \int_{\Omega} \frac{f(y)}{|x-y|^{n-2s+1}} dy. \quad (2.5)$$

The proof then follows as for  $s < 1/2$ , and one gets that

$$\begin{aligned} & |Du(x_1) - Du(x_2)| \\ & \leq \|f\|_{L^\infty(\mathbb{R}^n)} \left[ \int_{\Omega \cap \{|x_1-y| \leq 2\delta\}} \frac{dy}{|x_1-y|^{n-2s+1}} + \int_{\Omega \cap \{|x_1-y| \leq 2\delta\}} \frac{dy}{|x_2-y|^{n-2s+1}} \right. \\ & \quad \left. + \int_{\Omega \setminus \{|x_1-y| \leq 2\delta\}} \left| \frac{1}{|x_1-y|^{n-2s+1}} - \frac{1}{|x_2-y|^{n-2s+1}} \right| dy \right] \\ & \leq C\delta^{2s-1}, \end{aligned}$$

where  $C = C(n, s, f)$  is a positive constant. This concludes the proof of the Lemma.  $\square$

**Remark 2.2.** On  $\Omega$  one has the following bounds. For  $x_1, x_2 \in \Omega$

$$|u(x_1) - u(x_2)| \leq \begin{cases} C|x_1 - x_2| \left(1 + |x_1 - x_2|^{2s-1}\right) & \text{for } s < 1/2 \\ C|x_1 - x_2| \left(1 + |\ln |x_1 - x_2||\right) & \text{for } s = 1/2, \end{cases}$$

and

$$|Du(x_1) - Du(x_2)| \leq C|x_1 - x_2| \left(1 + |x_1 - x_2|^{2s-2}\right) \text{ for } s > 1/2,$$

where  $C = C(n, s, f, \Omega)$  is a positive constant depending on the dimension of the space, the fractional parameter  $s$ , the  $L^\infty$  norm of  $f$  and the diameter of  $\Omega$ .

To see these, it is enough to modify (2.3) as follows. Denoting  $R := \text{diam } \Omega$ , since  $|x_1| < R$  for  $s < 1/2$  we get that

$$I_3 \leq C_n \delta \int_{2\delta}^{2R} \rho^{2s-2} d\rho = C_{n,s} \delta (\delta^{2s-1} - R^{2s-1}) \leq C_{n,s} \delta (\delta^{2s-1} + 1).$$

For  $s = 1/2$ , we have  $I_1, I_2$  are bounded by  $C_{n,s} \delta$  and

$$I_3 \leq C_{n,s} \delta \ln \frac{R}{\delta} \leq C_{n,s,R} (1 + |\ln \delta|).$$

From this and the bounds established in the proof of Lemma 2.1, the estimates in this remark plainly follow.

We prove that the Riesz potential is  $C^{2s+\varepsilon}(\mathbb{R}^n)$  when  $f \in C_c^{0,\varepsilon}(\mathbb{R}^n)$ . The proof is included for completeness.

**Lemma 2.3.** Let  $s \in (0, 1)$  be fixed. Let  $f \in C_c^{0,\varepsilon}(\mathbb{R}^n)$  be a given function and  $u$  be defined as

$$u(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2s}} dy. \quad (2.6)$$

Then  $u \in C^{2s+\varepsilon}(\mathbb{R}^n)$ .

*Proof.* Let  $R > 0$  such that  $\text{supp } f \subseteq B_R$  and let  $s < 1/2$ . Then taking  $x_1, x_2 \in \mathbb{R}^n$  and denoting by  $\delta := |x_1 - x_2|$  we have that

$$\begin{aligned} & |u(x_1) - u(x_2)| \\ & \leq \int_{B_R \cap \{|x_1 - y| \leq 2\delta\}} \frac{|f(y) - f(x_1)|}{|x_1 - y|^{n-2s}} dy + \int_{B_R \cap \{|x_1 - y| \leq 2\delta\}} \frac{|f(y) - f(x_1)|}{|x_2 - y|^{n-2s}} dy \\ & \quad + \int_{B_R \setminus \{|x_1 - y| \leq 2\delta\}} |f(y) - f(x_1)| \left| \frac{1}{|x_1 - y|^{n-2s}} - \frac{1}{|x_2 - y|^{n-2s}} \right| dy \\ & \quad + |f(x_1)| \left| \int_{B_R} \left( \frac{1}{|x_1 - y|^{n-2s}} - \frac{1}{|x_2 - y|^{n-2s}} \right) dy \right| \\ & = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Since  $f$  is Hölder continuous, we have that for  $C > 0$

$$|f(y) - f(x_1)| \leq C|y - x_1|^\varepsilon.$$

Noticing that in the next computations the constants may change value from line to line, we obtain that

$$\begin{aligned} I_1 & \leq C \int_{B_R \cap \{|x_1 - y| \leq 2\delta\}} |x_1 - y|^{-n+2s+\varepsilon} dy = C_{n,s} \delta^{2s+\varepsilon}, \\ I_2 & \leq C \int_{B_R \cap \{|x_1 - y| \leq 2\delta\}} |x_1 - y|^\varepsilon |x_2 - y|^{-n+2s} dy \leq C_{n,s} \delta^{2s+\varepsilon} \end{aligned}$$

since  $|x_2 - y| \leq 3\delta$  and for  $x^*$  on the segment  $x_1 x_2$ , recalling that  $s < 1/2$

$$I_3 \leq C\delta \int_{B_R \setminus \{|x_1 - y| \leq 2\delta\}} \frac{|x_1 - y|^\varepsilon}{|x^* - y|^{n-2s+1}} dy \leq C_{n,s} \delta^{2s+\varepsilon},$$

given that  $|x^* - y| \geq \frac{1}{2}|x_1 - y|$ .

Now, if  $x_1 \notin B_R$  or  $x_2 \notin B_R$  (it is enough in this latter case to replace  $x_1$  with  $x_2$  in the above computations), then we are done. Else, for  $x_1, x_2 \in B_R$ , suppose that  $\text{dist}(x_1, \partial B_R) \geq \text{dist}(x_2, \partial B_R)$  and take  $p \in \partial B_R$  (hence  $f(p) = 0$ ) such that  $\text{dist}(x_1, \partial B_R) = |x_1 - p|$ . So

$$|f(x_1)| = |f(x_1) - f(p)| \leq C|x_1 - p|^\varepsilon$$

and we distinguish two cases. When

$$(1) \quad |x_1 - x_2| \geq \frac{1}{2}|x_1 - p|$$

we use the result in Lemma 2.1, observing that from (2.2) and (2.4) we have that

$$\int_{B_R} \left| \frac{1}{|x_1 - y|^{n-2s}} - \frac{1}{|x_2 - y|^{n-2s}} \right| dy \leq C|x_1 - x_2|^{2s},$$

hence

$$I_4 \leq C|x_1 - p|^\varepsilon |x_1 - x_2|^{2s} \leq C\delta^{2s+\varepsilon}.$$

On the other hand, when

$$(2) \quad |x_1 - x_2| \leq \frac{1}{2}|x_1 - p|$$

we use the following bound (see Lemmas 2.1 and 3.5 in [5])

$$\left| \int_{B_R} |x_1 - y|^{2s-n} - |x_2 - y|^{2s-n} dy \right| \leq \frac{|x_1 - x_2|}{\max\{\text{dist}(x_1, \partial B_R), \text{dist}(x_2, \partial B_R)\}^{1-2s}}. \quad (2.7)$$

Since  $2s + \varepsilon - 1 < 0$  we get that

$$I_4 \leq C\delta |x_1 - p|^{2s-1+\varepsilon} \leq C_{n,s} \delta^{2s+\varepsilon}.$$

This concludes the proof of the Lemma for  $s < 1/2$ . In order to prove the result for  $s \geq 1/2$ , one considers the formula (2.5) and iterates the computations above.  $\square$

The interested reader can see Theorem 4.6 in [5], where the result given here in Lemma 2.3 is proved for  $u$  defined as

$$u(x) = \int_{\Omega} \frac{f(y)}{|x-y|^{n-2s}} dy,$$

where  $\Omega$  is a domain with the  $s$ -property (see Definition 3.3 therein). In particular, these domains are defined such that they satisfy a bound of the type given in (2.7), while the ball is the typical example of this type.

### 3. SOME USEFUL ESTIMATES

In this section we introduce some useful estimates, using the representation formulas (1.4) and (1.6). The interested reader can also check [8], where Cauchy-type estimates for the derivatives of  $s$ -harmonic functions are proved using the Riesz and Poisson kernel.

We fix  $r > 0$ .

**Lemma 3.1.** *Let  $u \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n \setminus B_r)$  be such that  $(-\Delta)^s u(x) = 0$  for any  $x$  in  $B_r$ . Then for any  $\alpha \in \mathbb{N}_0^n$*

$$\|D^\alpha u\|_{L^\infty(B_{r/2})} \leq cr^{-|\alpha|} \|u\|_{L^\infty(\mathbb{R}^n \setminus B_r)}, \quad (3.1)$$

where  $c = c(n, s, \alpha)$  is a positive constant.

*Proof.* We notice that it is enough to prove (3.1) for  $r = 1$ , i.e.

$$\|D^\alpha u\|_{L^\infty(B_{1/2})} \leq c \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)}. \quad (3.2)$$

Indeed, if (3.2) holds, then by rescaling, namely letting  $y = rx$  and  $v(y) = u(x)$  for  $x \in B_1$ , we have that  $D^\alpha u(x) = r^{|\alpha|} D^\alpha v(y)$ . Hence  $r^{|\alpha|} |D^\alpha v(y)| = |D^\alpha u(x)| \leq c \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)} = c \|v\|_{L^\infty(\mathbb{R}^n \setminus B_r)}$  and one gets the original estimate for any  $r$ .

We use the representation formula given in (1.4). By inserting definition (1.3), we have that in  $B_1$

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n \setminus B_1} u(y) P_1(y, x) dy \\ &= c(n, s) \int_{\mathbb{R}^n \setminus B_r} u(y) \frac{(1 - |x|^2)^s}{(|y|^2 - 1)^s} \frac{dy}{|x - y|^n}. \end{aligned}$$

Let  $x \in B_{1/2}$ . We take the  $j^{th}$  derivative of  $u$  and have that

$$\begin{aligned} D_j u(x) &= c(n, s) \int_{\mathbb{R}^n \setminus B_1} u(y) D_j \left[ \frac{(1 - |x|^2)^s}{(|y|^2 - 1)^s} \frac{1}{|x - y|^n} \right] dy \\ &= c(n, s) \int_{\mathbb{R}^n \setminus B_r} \frac{u(y)}{(|y|^2 - 1)^s} \left[ \frac{(-2sx_j)(1 - |x|^2)^{s-1}}{|x - y|^n} + (-n) \frac{(1 - |x|^2)^s (x_j - y_j)}{|x - y|^{n+2}} \right] dy. \end{aligned}$$

Therefore renaming the constants (even from line to line),

$$|Du(x)| \leq c_{n,s} \int_{\mathbb{R}^n \setminus B_1} \frac{|u(y)|}{(|y|^2 - 1)^s} \left[ \frac{|x|(1 - |x|^2)^{s-1}}{|x - y|^n} + \frac{(1 - |x|^2)^s}{|x - y|^{n+1}} \right] dy. \quad (3.3)$$

Given that  $|x| \leq 1/2$  we have that  $3/4 \leq 1 - |x|^2 \leq 1$  and  $|x - y| \geq |y|/2$  and so

$$|Du(x)| \leq c_{n,s} \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)} \int_{\mathbb{R}^n \setminus B_1} \left[ \frac{1}{(|y| - 1)^s |y|^n} + \frac{1}{(|y| - 1)^s |y|^{n+1}} \right] dy.$$



Passing to polar coordinates and renaming the constants, we have that

$$|Du(x)| \leq c_{n,s} \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)} \left[ \int_1^\infty (\rho-1)^{-s} \rho^{-1} d\rho + \int_1^\infty (\rho-1)^{-s} \rho^{-2} d\rho \right].$$

Now we compute

$$\int_1^\infty (\rho-1)^{-s} \rho^{-1} d\rho = \int_1^2 (\rho-1)^{-s} \rho^{-1} d\rho + \int_2^\infty (\rho-1)^{-s} \rho^{-1} d\rho \leq C$$

and likewise,

$$\int_1^\infty (\rho-1)^{-s} \rho^{-2} d\rho \leq C.$$

It follows that

$$|Du(x)| \leq c_{n,s} \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)} \quad \text{for any } x \in B_{1/2}.$$

By reiterating the computation, we obtain the conclusion for the  $\alpha$  derivative. This proves the estimate (3.2), thus (3.1) by rescaling.  $\square$

**Lemma 3.2.** *Let  $f \in C^{0,\varepsilon}(B_r) \cap C(\overline{B_r})$  be a given function and  $u \in C^1(B_r) \cap L^\infty(\mathbb{R}^n)$  be a pointwise solution of*

$$\begin{cases} (-\Delta)^s u = f & \text{in } B_r, \\ u = 0 & \text{in } \mathbb{R}^n \setminus B_r. \end{cases}$$

Then

$$\|u\|_{L^\infty(B_r)} \leq cr^{2s} \sup_{\overline{B_r}} |f|, \quad (3.4)$$

where  $c = c(n, s)$  is a positive constant. Furthermore, for  $s > 1/2$

$$\|Du\|_{L^\infty(B_{r/2})} \leq \bar{c}r^{2s-1} \sup_{\overline{B_r}} |f|, \quad (3.5)$$

where  $\bar{c} = \bar{c}(n, s)$  is a positive constant.

*Proof.* We notice that it is enough to prove (3.4) and (3.5) for  $r = 1$ , i.e.

$$\|u\|_{L^\infty(B_1)} \leq c \sup_{\overline{B_1}} |f| \quad (3.6)$$

and

$$\|Du\|_{L^\infty(B_{1/2})} \leq \bar{c} \sup_{\overline{B_1}} |f|. \quad (3.7)$$

Indeed, by rescaling, we let  $y = rx$  and  $v(y) = u(x)$  we have that  $(-\Delta)^s v(y) = r^{-2s} (-\Delta)^s u(x)$ , while  $rDv(y) = Du(x)$  and one gets the original estimates for any  $r$ .

We take  $\tilde{f}$  to be a continuous extension of  $f$ , namely let  $\tilde{f} \in C_c^{0,\varepsilon}(\mathbb{R}^n)$  be such that

$$\tilde{f} = \begin{cases} f & \text{in } B_1 \\ 0 & \text{in } \mathbb{R}^n \setminus B_{3/2} \end{cases}$$

and  $\sup_{\mathbb{R}^n} |\tilde{f}| \leq C \sup_{\overline{B_1}} |f|$ . Let

$$\tilde{u}(x) := \tilde{f} * \Phi(x) = a(n, s) \int_{\mathbb{R}^n} \frac{\tilde{f}(y)}{|x-y|^{n-2s}} dy. \quad (3.8)$$

Then  $\tilde{u} \in L_s^1(\mathbb{R}^n)$  (see Theorem 2.3 in [1] for the proof) and  $\tilde{u} \in C^{2s+\varepsilon}(\mathbb{R}^n)$ , according to Lemma 2.3. Thanks to (1.7), we have that  $(-\Delta)^s \tilde{u} = \tilde{f}$  pointwise in  $\mathbb{R}^n$ . Hence, thanks to the definition of  $\tilde{f}$ ,  $(-\Delta)^s (\tilde{u} - u) = 0$  in  $B_1$ . Moreover,  $\tilde{u} - u = \tilde{u}$  in  $\mathbb{R}^n \setminus B_1$  and from (1.4) we have that in  $B_1$

$$(\tilde{u} - u)(x) = \int_{\mathbb{R}^n \setminus B_1} \tilde{u}(y) P_1(y, x) dy. \quad (3.9)$$

We notice at first that by definition (3.8) and passing to polar coordinates, we obtain for any positive constant  $\tilde{c}$  that

$$\|\tilde{u}\|_{L^\infty(B_{\tilde{c}})} \leq a_{n,s} \sup_{\mathbb{R}^n} |\tilde{f}| \int_0^{\tilde{c}+3/2} \rho^{2s-1} d\rho \leq c_{n,s} \sup_{\overline{B}_1} |f|. \quad (3.10)$$

By renaming constants, we also have that

$$\begin{aligned} \|\tilde{u} - u\|_{L^\infty(B_1)} &\leq \int_{B_2 \setminus B_1} |\tilde{u}(y)| P_1(y, x) dy + \int_{\mathbb{R}^n \setminus B_2} |\tilde{u}(y)| P_1(y, x) dy \\ &\leq \|\tilde{u}\|_{L^\infty(B_2)} + I \\ &\leq c_{n,s} \sup_{\overline{B}_1} |f| + I. \end{aligned} \quad (3.11)$$

Inserting the definition (1.3) and using for  $|y| \geq 2$  the bounds  $|y-x| \geq |y|/2$  and  $|y|^2 - 1 \geq |y|^2/2$  we have that

$$\begin{aligned} I &\leq c(n, s) \int_{\mathbb{R}^n \setminus B_2} \frac{|\tilde{u}(y)|}{(|y|^2 - 1)^s |x - y|^n} dy \\ &\leq c_{n,s} \int_{\mathbb{R}^n \setminus B_2} \frac{|\tilde{u}(y)|}{|y|^{n+2s}} dy. \end{aligned}$$

We estimate the  $L_s^1$  norm of  $\tilde{u}$  as follows

$$\begin{aligned} \|\tilde{u}\|_{L_s^1(\mathbb{R}^n \setminus B_2)} &= \int_{\mathbb{R}^n \setminus B_2} \frac{|\tilde{u}(y)|}{|y|^{n+2s}} dy \\ &\leq a(n, s) \int_{\mathbb{R}^n \setminus B_2} |y|^{-n-2s} \left[ \int_{B_{3/2}} \frac{|\tilde{f}(t)|}{|y-t|^{n-2s}} dt \right] dy \\ &\leq a(n, s) \sup_{\mathbb{R}^n} |\tilde{f}| \int_{B_{3/2}} \left[ \int_{\mathbb{R}^n \setminus B_2} |y|^{-n-2s} |y-t|^{2s-n} dy \right] dt. \end{aligned} \quad (3.12)$$

We use that  $|y-t| \geq |y|/4$  and passing to polar coordinates we get that

$$\|\tilde{u}\|_{L_s^1(\mathbb{R}^n \setminus B_2)} \leq a_{n,s} \sup_{\mathbb{R}^n} |\tilde{f}| \int_2^\infty \rho^{-n-1} d\rho = a_{n,s} \sup_{\overline{B}_1} |f|. \quad (3.13)$$

Hence

$$I \leq c_{n,s} \sup_{\overline{B}_1} |f|.$$

It follows in (3.11) (eventually renaming the constants) that

$$\|\tilde{u} - u\|_{L^\infty(B_1)} \leq c_{n,s} \sup_{\overline{B}_1} |f|. \quad (3.14)$$

By the triangle inequality, we have that

$$\|u\|_{L^\infty(B_1)} \leq \|\tilde{u}\|_{L^\infty(B_1)} + \|\tilde{u} - u\|_{L^\infty(B_1)}.$$

Hence by using (3.10) and (3.14) we obtain that

$$\|u\|_{L^\infty(B_1)} \leq c_{n,s} \sup_{\overline{B}_1} |f|,$$

that is the desired estimate (3.6), hence (3.4) after rescaling.

In order to prove (3.7), we take  $x \in B_{1/2}$  and obtain by the triangle inequality that

$$|Du(x)| \leq |D(\tilde{u} - u)(x)| + |D\tilde{u}(x)|. \quad (3.15)$$

We notice that in the next computations the constants may change value from line to line. By using (3.9) and (3.3), for  $|x| \leq 1/2$  (hence  $|y - x| \geq |y|/2$ ) we obtain that

$$\begin{aligned} |D(\tilde{u} - u)(x)| &\leq c_{n,s} \int_{\mathbb{R}^n \setminus B_1} \frac{|\tilde{u}(y)|}{(|y|^2 - 1)^s |y|^n} dy \\ &\quad + c_{n,s} \int_{\mathbb{R}^n \setminus B_1} \frac{|\tilde{u}(y)|}{(|y|^2 - 1)^s |y|^{n+1}} dy \\ &= c_{n,s} (I_1 + I_2). \end{aligned} \tag{3.16}$$

We compute by passing to polar coordinates that

$$\int_{B_2 \setminus B_1} \frac{|\tilde{u}(y)|}{(|y|^2 - 1)^s |y|^n} dy \leq c_{n,s} \|\tilde{u}\|_{L^\infty(B_2)} \leq c_{n,s} \sup_{\overline{B_1}} |f|,$$

according to (3.10). Moreover, for  $|y| \geq 2$  we have that  $|y|^2 - 1 \geq |y|^2/2$  and so

$$\int_{\mathbb{R}^n \setminus B_2} \frac{|\tilde{u}(y)|}{(|y|^2 - 1)^s |y|^n} dy \leq \int_{\mathbb{R}^n \setminus B_2} \frac{|\tilde{u}(y)|}{|y|^{2s+n}} dy \leq a_{n,s} \sup_{\overline{B_1}} |f|$$

thanks to (3.12) and (3.13). Hence

$$I_1 \leq c_{n,s} \sup_{\overline{B_1}} |f|.$$

We split also integral  $I_2$  into two and by passing to polar coordinates, we get that

$$\int_{B_2 \setminus B_1} \frac{|\tilde{u}(y)|}{(|y|^2 - 1)^s |y|^{n+1}} dy \leq c_{n,s} \|\tilde{u}\|_{L^\infty(B_2)} \leq c_{n,s} \sup_{\overline{B_1}} |f|$$

again by (3.10). Also, using definition (3.8) of  $\tilde{u}$  and for  $|y| \geq 2$  the fact that  $|y|^2 - 1 \geq |y|^2/2$ , we get

$$\int_{\mathbb{R}^n \setminus B_2} \frac{|\tilde{u}(y)|}{(|y|^2 - 1)^2 |y|^{n+1}} dy \leq a(n, s) \int_{\mathbb{R}^n \setminus B_2} |y|^{-n-2s-1} \left[ \int_{B_{3/2}} \frac{|\tilde{f}(t)|}{|y - t|^{n-2s}} dt \right] dy.$$

We have that  $|y - t| \geq |y|/4$  and obtain that

$$\int_{\mathbb{R}^n \setminus B_2} \frac{|\tilde{u}(y)|}{(|y|^2 - 1)^2 |y|^{n+1}} dy \leq a_{n,s} \sup_{\overline{B_1}} |f|.$$

It follows that

$$I_2 \leq c_{n,s} \sup_{\overline{B_1}} |f|.$$

Inserting the bounds on  $I_1$  and  $I_2$  into (3.16), we finally obtain that

$$|D(\tilde{u} - u)(x)| \leq c_{n,s} \sup_{\overline{B_1}} |f|. \tag{3.17}$$

On the other hand, for  $s > 1/2$ , using (2.5) we get that

$$D\tilde{u}(x) = a(n, s) \int_{B_{3/2}} \frac{\tilde{f}(y)}{|x - y|^{n-2s+1}} dy$$

and therefore by passing to polar coordinates

$$\begin{aligned} |D\tilde{u}(x)| &\leq a_{n,s} \sup_{\overline{B_1}} |f| \int_{B_{3/2}} |x - y|^{2s-n-1} dy \leq a_{n,s} \sup_{\overline{B_1}} |f| \int_0^2 \rho^{2s-2} d\rho \\ &= a_{n,s} \sup_{\overline{B_1}} |f|. \end{aligned}$$

This and (3.17) finally allow us to conclude from (3.15) that

$$|Du(x)| \leq \bar{c} \sup_{\bar{B}_1} |f|$$

for any  $x \in B_{1/2}$ , therefore the bound in (3.7). From this after rescaling, we obtain the estimate in (3.5).  $\square$

#### 4. A PROOF OF SCHAUDER ESTIMATES

In this section we give a simple proof of some Schauder estimates related to the fractional Laplacian, as stated in Theorem 1.2. As we see by substituting in (1.1) and (1.2) that  $\omega(r) \leq Cr^\alpha$ , we obtain for  $s \leq 1/2$

$$|u(x) - u(y)| \leq C_{n,s} \delta \left( \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)} + \sup_{\bar{B}_1} |f| + \delta^{\alpha+2s-1} \right),$$

hence  $u \in C^{0,2s+\alpha}(B_{1/2})$  as long as  $\alpha < 1 - 2s$  and Lipschitz if  $\alpha > 1 - 2s$ . For  $s > 1/2$  we have that

$$|Du(x) - Du(y)| \leq C_{n,s} \delta \left( \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)} + \sup_{\bar{B}_1} |f| + \delta^{\alpha+2s-2} \right).$$

Hence if  $\alpha \leq 2 - 2s$  then  $u \in C^{1,\alpha+2s-1}(B_{1/2})$  while for  $2 - 2s \leq \alpha < 1$  the derivative  $Du$  is Lipschitz in  $B_{1/2}$ . The proof takes its inspiration from [14], where a similar result is proved for the classical case of the Laplacian.

We prove here the case  $s > 1/2$ , noting that for  $s \leq 1/2$  the proof follows in the same way, using the lower order estimates.

*Proof of Theorem 1.2.* For  $k = 1, 2, \dots$ , we denote by  $B_k := B_{\rho^k}(0)$ , where  $\rho = 1/2$  and let  $u_k$  be a solution of

$$\begin{cases} (-\Delta)^s u_k = f(0) & \text{in } B_k \\ u_k = u & \text{in } \mathbb{R}^n \setminus B_k. \end{cases}$$

Then we have that

$$\begin{cases} (-\Delta)^s (u_k - u) = f(0) - f & \text{in } B_k \\ u_k - u = 0 & \text{in } \mathbb{R}^n \setminus B_k. \end{cases}$$

We remark that in the next computations, the constants may change value from line to line.

Thanks to (3.4), we get that

$$\begin{aligned} \|u_k - u\|_{L^\infty(B_k)} &\leq c_{n,s} \rho^{2ks} \sup_{B_k} |f(0) - f| \\ &\leq c_{n,s} \rho^{2ks} \omega(\rho^k). \end{aligned} \tag{4.1}$$

Using (3.5), we obtain that

$$\|D(u_k - u)\|_{L^\infty(B_{k+1})} \leq c_{n,s} \rho^{(2s-1)k} \omega(\rho^k). \tag{4.2}$$

From here, sending  $k$  to infinity, for  $s > 1/2$  it yields that

$$\lim_{k \rightarrow \infty} Du_k(0) = Du(0). \tag{4.3}$$

Furthermore,

$$\begin{cases} (-\Delta)^s (u_k - u_{k+1}) = 0 & \text{in } B_{k+1} \\ u_k - u_{k+1} = u_k - u & \text{in } B_k \setminus B_{k+1} \\ u_k - u_{k+1} = 0 & \text{in } \mathbb{R}^n \setminus B_k, \end{cases}$$

hence from (3.1) we have that

$$\|D(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \leq c_{n,s} \rho^{-(k+1)} \sup_{B_k \setminus B_{k+1}} |u_k - u|$$

and

$$\|D^2(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \leq c_{n,s} \rho^{-2(k+1)} \sup_{B_k \setminus B_{k+1}} |u_k - u|.$$

Using now (4.1), we get that

$$\|D(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \leq c_{n,s} \rho^{(2s-1)k} \omega(\rho^k) \quad (4.4)$$

and

$$\|D^2(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \leq c_{n,s} \rho^{(2s-2)k} \omega(\rho^k). \quad (4.5)$$

Let us fix  $s > 1/2$ . Then for any given point  $z$  near the origin we have that

$$\begin{aligned} |Du(z) - Du(0)| &\leq |Du_k(z) - Du(z)| + |Du_k(0) - Du(0)| + |Du_k(z) - Du_k(0)| \\ &= A_1 + A_2 + A_3. \end{aligned} \quad (4.6)$$

For  $k \in \mathbb{N}^*$  fixed, we take  $z$  such that  $\rho^{k+2} \leq |z| \leq \rho^{k+1}$ . Using (4.2) we get that

$$A_1 \leq c_{n,s} \rho^{(2s-1)k} \omega(\rho^k).$$

Taking into account (4.3) and using (4.4), we have that

$$A_2 \leq \sum_{j=k}^{\infty} |Du_j(0) - Du_{j+1}(0)| \leq c_{n,s} \sum_{j=k}^{\infty} \rho^{(2s-1)j} \omega(\rho^j),$$

therefore by renaming the constants

$$\begin{aligned} A_1 + A_2 &\leq c_{n,s} \rho^{(2s-1)k} \omega(\rho^k) + c_{n,s} \sum_{j=k}^{\infty} \rho^{(2s-1)j} \omega(\rho^j) \\ &\leq c_{n,s} \sum_{j=k}^{\infty} \rho^{(2s-1)j} \omega(\rho^j). \end{aligned}$$

For the positive constant  $c_s = (2s-1)/(\rho^{1-2s} - 1)$  and any  $j = k, k+1, \dots$  we have that

$$\rho^{(2s-1)j} = c_s \int_{\rho^j}^{\rho^{j-1}} t^{2s-2} dt.$$

Since  $\omega$  is a increasing function, we obtain that

$$\omega(\rho^j) \rho^{(2s-1)j} = c_s \omega(\rho^j) \int_{\rho^j}^{\rho^{j-1}} t^{2s-2} dt \leq c_s \int_{\rho^j}^{\rho^{j-1}} \omega(t) t^{2s-2} dt,$$

hence given that  $8|z| \geq \rho^{k-1}$

$$\begin{aligned} \sum_{j=k}^{\infty} \rho^{(2s-1)j} \omega(\rho^j) &\leq c_s \sum_{j=k}^{\infty} \int_{\rho^j}^{\rho^{j-1}} \omega(t) t^{2s-2} dt \leq c_s \int_0^{\rho^{k-1}} \omega(t) t^{2s-2} dt \\ &\leq c_s \int_0^{8|z|} \omega(t) t^{2s-2} dt. \end{aligned}$$

Therefore,

$$A_1 + A_2 \leq c_{n,s} \int_0^{8|z|} \omega(t) t^{2s-2} dt. \quad (4.7)$$

Moreover, for  $j = 0, 1, \dots, k-1$  we consider  $h_j := u_{j+1} - u_j$  and have that

$$A_3 \leq \sum_{j=0}^{k-1} |Dh_j(z) - Dh_j(0)| + |Du_0(z) - Du_0(0)|.$$

By the mean value theorem, there exists  $\theta \in (0, |z|)$  such that

$$|Dh_j(z) - Dh_j(0)| \leq |z| |D^2 h_j(\theta)|$$

and since  $|z| \leq \rho^{k+1}$ , thanks to (4.5) we obtain that

$$|D^2 h_j(\theta)| \leq c_{n,s} \rho^{(2s-2)j} \omega(\rho^j).$$

Hence

$$\sum_{j=0}^{k-1} |Dh_j(z) - Dh_j(0)| \leq c_{n,s} |z| \sum_{j=0}^{k-1} \rho^{(2s-2)j} \omega(\rho^j) = c_{n,s} |z| \left( \sup_{\overline{B}_1} |f| + \sum_{j=1}^{k-1} \rho^{(2s-2)j} \omega(\rho^j) \right).$$

As previously done, we have that for the positive constant  $c_s = (2 - 2s) / (1 - \rho^{2-2s})$  and  $j = 1, \dots, k-1$

$$\rho^{(2s-2)j} = c_s \int_{\rho^j}^{\rho^{j-1}} t^{2s-3} dt$$

and since  $\omega$  is increasing

$$\omega(\rho^j) \rho^{(2s-2)j} \leq c_s \int_{\rho^j}^{\rho^{j-1}} \omega(t) t^{2s-3} dt.$$

It follows that

$$\begin{aligned} \sum_{j=1}^{k-1} \rho^{(2s-2)j} \omega(\rho^j) &\leq c_s \sum_{j=1}^{k-1} \int_{\rho^j}^{\rho^{j-1}} \omega(t) t^{2s-3} dt \leq c_s \int_{\rho^{k-1}}^1 \omega(t) t^{2s-3} dt \\ &\leq c_s \int_{|z|}^1 \omega(t) t^{2s-3} dt, \end{aligned}$$

since  $|z| \leq \rho^{k-1}$ . Therefore,

$$\sum_{j=1}^{k-1} |Dh_j(z) - Dh_j(0)| \leq c_{n,s} |z| \int_{|z|}^1 \omega(t) t^{2s-3} dt.$$

Moreover, let

$$v_0(x) := k_{n,s} f(0) (1 - |x|^2)_+^s \quad \text{for } x \in \mathbb{R}^n.$$

Then (see Section 2.6 in [2], or the general result in [7]), for the appropriate value of  $k_{n,s}$ , we have in  $B_1$  that  $(-\Delta)^s v_0(x) = f(0)$ . Then the function  $u_0 - v_0$  is  $s$ -harmonic in  $B_1$ , with boundary data  $u$ . We have that

$$|Du_0(z) - Du_0(0)| \leq |z| |D^2 u_0(\theta)| \leq |z| (|D^2(u_0 - v_0)(\theta)| + |D^2 v_0(\theta)|).$$

Using the estimate in (3.1) we have for  $\theta \in (0, |z|)$

$$|D^2(u_0 - v_0)(\theta)| \leq c_{n,s} \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)}.$$

Moreover,  $|D^2 v_0(\theta)|$  is bounded. It follows that

$$|Du_0(z) - Du_0(0)| \leq c_{n,s} |z| \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)},$$

hence

$$A_3 \leq c_{n,s} |z| \left( \sup_{\overline{B}_1} |f| + \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)} + \int_{|z|}^1 \omega(t) t^{2s-3} dt \right).$$

Inserting this and (4.7) into (4.6) we finally obtain that

$$|Du(z) - Du(0)| \leq C_{n,s} \left[ |z| \left( \|u\|_{L^\infty(\mathbb{R}^n \setminus B_1)} + \sup_{\overline{B_1}} |f| \right) + \int_0^{c|z|} \omega(t) t^{2s-2} dt \right. \\ \left. + |z| \int_{|z|}^1 \omega(t) t^{2s-3} dt \right].$$

From this the conclusion plainly follows. This concludes the proof of the Theorem.  $\square$

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